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Generalized Gagliardo–Nirenberg estimates and differentiability of the solutions to monotone nonlinear parabolic systems

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Abstract In this paper, we are concerned with interior differentiability of weak solutions u to nonlinear parabolic systems with natural growth and coefficients uniformly monotone in Du. Making use of estimates of Gagliardo–Nirenberg's type in generalized Sobolev spaces, we show that u belongs to $L^2(-a, 0, H^2(B(\sigma), \mathbb{R}^N)) \cap H^1(-a, 0, L^2(B(\sigma), \mathbb{R}^N))$ (see Theorem 3).

Keywords Generalized Sobolev spaces · Gagliardo–Nirenberg estimates · Nonlinear parabolic systems · Interior differentiability

1 Introduction

Let Ω be an open bounded subset of $\mathbb{R}^n$ $(n > 2)$ of generic point $x = (x_1, x_2, \ldots, x_n)$ and let Q be the cylinder $\Omega \times (-T, 0)$ $(0 < T < +\infty)$.

Let us consider a weak solution $u = (u_1, u_2, \ldots, u_N) \in L^2(-T, 0, H^1(\Omega, \mathbb{R}^N)) \bigcap C^{0,\lambda}(\overline{Q}, \mathbb{R}^N)$ (N positive integer, $0 < \lambda < 1$) to the second order nonlinear parabolic system of variational type with quadratic growth[1]

$$-\sum_{i=1}^{n} D_i a^i(X, u, Du) + \frac{\partial u}{\partial t} = B^0(X, u, Du) \tag{1}$$

[1] For notation and symbols we refer to [2]. In particular, if u: $Q \to \mathbb{R}^N$, we shall write $Du = (D_1 u| D_2 u| \cdots |D_n u)$, $D_i = \frac{\partial}{\partial x_i}$.

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in the sense that

$$\int_Q \left\{ \sum_{i=1}^{n} (a^i(X,u,Du)|D_i\varphi) - \left(u \middle| \frac{\partial \varphi}{\partial t}\right) \right\} \mathrm{d}X$$
$$= \int_Q (B^0(X,u,Du)|\varphi)\mathrm{d}X, \quad \forall \varphi \in C_0^\infty(Q,\mathbb{R}^N), \tag{2}$$

where $X = (x,t)$ and $a^i(X,u,p)$, $i = 1,2,\ldots,n$, and $B^0(X,u,p)$ are vectors of $\mathbb{R}^N$ defined on $\Lambda = \overline{Q} \times \mathbb{R}^N \times \mathbb{R}^{nN}$, measurable in X, continuous in (u,p) and satisfying the following conditions:

(i) *There exist two positive constants $M(K)$ and $\nu(K)$ such that* [2]

$$\|a(X,u,p) - a(X,u,\overline{p})\| \le M(K)\|p - \overline{p}\|,$$

$$\left(a(X,u,p) - a(X,u,\overline{p}) \middle| p - \overline{p}\right) \ge \nu(K)\|p - \overline{p}\|^2$$

for all $(X,u) \in Q \times \mathbb{R}^N$, with $\|u\| \le K$, and all $p, \overline{p} \in \mathbb{R}^{nN}$;

(ii) *For every $x \in \Omega$, $y \in B\left(x, \frac{1}{\sqrt{2}}d_x\right)$ $(d_x = \mathrm{dist}(\{x\},\partial\Omega))$, $t \in (-T,0)$, $u,v \in \mathbb{R}^N$ with $\|u\|, \|v\| \le K$, $p \in \mathbb{R}^{nN}$ one has*

$$\|a(X,u,p)\| \le M(K)(1 + \|p\|),$$

$$\|a(x,t,u,p) - a(y,t,v,p)\| \le M(K)(\|x-y\| + \|u-v\|)(1 + \|p\|);$$

(iii) *For every $(X,u,p) \in \Lambda$ with $\|u\| \le K$, one has*

$$\|B^0(X,u,p)\| \le M(K)(1 + \|p\|^2).$$

For many years an open problem has been the following:
Under the above assumptions, show that the weak solutions

$$u \in L^2(-T,0,H^1(\Omega,\mathbb{R}^N)) \cap C^{0,\lambda}(\overline{Q},\mathbb{R}^N)$$

to system (1) are locally differentiable, namely

$$u \in L^2(-a,0,H^2(B(\sigma),\mathbb{R}^N)) \cap H^1(-a,0,L^2(B(\sigma),\mathbb{R}^N))$$

for each cube $B(3\sigma) = B(x^0,3\sigma) = \{x \in \mathbb{R}^n : |x_i - x_i^0| < 3\sigma,\ i = 1,2,\ldots,n\} \subset\subset \Omega$ and $\forall a \in (0,T)$.

A partial answer to this question was given by Fattorusso [3] (see also [2]), who proved the following result:

For each cube $B(\sigma) \subset\subset B(\sigma_0) \subset\subset \Omega$ and each $a, b \in (0,T)$, with $a < b$, for the weak solutions to system (1), we have:

$$u \in L^2(-a,0,H^{1+\vartheta}(B(\sigma),\mathbb{R}^N)), \quad \forall \vartheta \in (0,1), \tag{3}$$

[2] $a(X,u,p) = (a^1(X,u,p)|\cdots|a^n(X,u,p)) \in \mathbb{R}^{nN}$.

and the following estimate[3]

$$\int_{-a}^{0} |Du|^2_{\vartheta, B(\sigma)} \mathrm{d}t \le c \left\{ \int_{-b}^{0} |u|^2_{1, B(\sigma_0)} \mathrm{d}t + 1 \right\}$$

holds.

From this result, we can derive that the weak solutions u of system (1) belong to the space $L^2(-a, 0, H^2(B(\sigma), \mathbb{R}^N))$, once we are able to show the further regularity result:

$$D_i u \in L^4(B(\sigma) \times (-a, 0), \mathbb{R}^N), \quad i = 1, 2, \dots, n.$$

We justify our claim, by making two remarks.

(j) If $u \in L^2(-T, 0, H^1(\Omega, \mathbb{R}^N)) \cap C^{0,\lambda}(\overline{Q}, \mathbb{R}^N)$, $0 < \lambda < 1$, is a solution of system (2) and if hypotheses *(i)*, *(ii)*, and *(iii)* hold, then the following inequality holds true (see (3.8) of [3]):

$$\begin{aligned} &\frac{\nu}{2} \int_{-b}^{-\frac{1}{m}} \mathrm{d}t \int_{B((\sigma+3\sigma_0)/4)} \psi^2 \rho_m^2 \|\tau_{i,h} Du\|^2 \mathrm{d}x \\ &\le c(\nu, K, \sigma, \sigma_0, a, b, n) h^2 \int_{-b}^{-\frac{1}{m}} \mathrm{d}t \int_{B(\sigma_0)} (1 + \|Du\|^2) \mathrm{d}x \\ &\quad + c(\nu, K) \int_{-b}^{-\frac{1}{m}} \mathrm{d}t \int_{B((\sigma+3\sigma_0)/4)} \psi^2 \rho_m^2 \|\tau_{i,h} u\|^2 \|Du\|^2 \mathrm{d}x \\ &\quad + c(K) \int_{-b}^{-\frac{1}{m}} \rho_m^2 \mathrm{d}t \int_{B((\sigma+7\sigma_0)/8)} (1 + \|Du\|^2) \|\tau_{i,-h}(\psi^2 \tau_{i,h} u)\| \mathrm{d}x, \end{aligned} \tag{4}$$

where $B(\sigma) \subset\subset B(\sigma_0) \subset\subset \Omega$, $a, b \in (0, T)$ with $a < b$, $h \in \mathbb{R}$ with $|h| < \frac{\sigma_0 - \sigma}{8}$, $i \in \mathbb{N}$ with $1 \le i \le n$, $\tau_{i,\pm h} v(X) = v(x \pm he^i, t) - v(X)$, $\psi(x)$ is a real function of class $C_0^\infty(\mathbb{R}^n)$ having the following properties:

$$\begin{aligned} &0 \le \psi \le 1, \quad \psi = 1 \text{ in } B\left(\frac{\sigma + \sigma_0}{2}\right), \quad \psi = 0 \text{ in } \mathbb{R}^n \setminus B\left(\frac{\sigma + 3\sigma_0}{4}\right), \\ &\|D\psi\| \le \frac{c}{\sigma_0 - \sigma} \end{aligned} \tag{5}$$

[3] If $1 < r < \infty$, $0 < \vartheta < 1$ and $m = 0, 1, 2, \dots$, we shall set

$$|u|_{\vartheta,r,\Omega} = \left(\int_\Omega \mathrm{d}x \int_\Omega \frac{\|u(x) - u(y)\|^r}{\|x - y\|^{n+\vartheta r}} \mathrm{d}y \right)^{1/r}, \quad |u|_{m,r,\Omega} = \left(\int_\Omega \sum_{|\alpha|=m} \|D^\alpha u\|^r \,\mathrm{d}x \right)^{1/r},$$

$$\|u\|_{m,r,\Omega} = \left(\sum_{k=0}^{m} |u|^r_{k,r,\Omega} \right)^{1/r}, \quad \|u\|_{m+\vartheta,r,\Omega} = \left(\|u\|^r_{m,r,\Omega} + \sum_{|\alpha|=m} |D^\alpha u|^r_{\vartheta,r,\Omega} \right)^{1/r}.$$

For the sake of simplicity we shall write: $|\cdot|_{\vartheta,\Omega}$, $|\cdot|_{m,\Omega}$, $\|\cdot\|_{m,\Omega}$, $\|\cdot\|_{m+\vartheta,\Omega}$ instead of $|\cdot|_{\vartheta,2,\Omega}$, $|\cdot|_{m,2,\Omega}$, $\|\cdot\|_{m,2,\Omega}$, $\|\cdot\|_{m+\vartheta,2,\Omega}$, respectively, and $|Du|^2_{\vartheta,\Omega} = \sum_{i=1}^{n} |D_i u|^2_{\vartheta,\Omega}$.

and $\rho_m(t)$, m integer $> 2/a$, is the real function defined onto $\mathbb{R}$ by setting:

$$\begin{aligned} &\rho_m(t) = 1 \quad \text{if } -a \le t \le -2/m, \\ &\rho_m(t) = 0 \quad \text{if } t \ge -1/m \text{ and } t \le -b, \\ &\rho_m(t) = \frac{t+b}{b-a} \quad \text{if } -b < t < -a \\ &\rho_m(t) = -(mt+1) \quad \text{if } -2/m < t < -1/m. \end{aligned} \tag{6}$$

It is now clear that from (4) we can deduce:

$$u \in L^2(-a, 0, H^2(B(\sigma), \mathbb{R}^N)) \tag{7}$$

if we are able to estimate the last two integrals of the right-hand side by means of the Hölder inequality. This is possible if we know that

$$Du \in L^4(B(\sigma_0) \times (-b, 0), \mathbb{R}^{nN}). \tag{8}$$

(jj) The following result, due to Nirenberg [13], is well known:
If $u \in H^2(\Omega, \mathbb{R}^N) \bigcap C^{0,\lambda}(\Omega, \mathbb{R}^N)$, with Ω cube of $\mathbb{R}^n$, N positive integer and $0 < \lambda < 1$, then there exist two constants c_1 and c_2 (depending on Ω and λ) such that[4]

$$|Du|_{0,p,\Omega} \le c_1 |D^2 u|^a_{0,2,\Omega} [u]^{1-a}_{\lambda,\Omega} + c_2 [u]_{\lambda,\Omega}, \tag{9}$$

where $\dfrac{1}{p} = \dfrac{1}{n} + a\left(\dfrac{1}{2} - \dfrac{2}{n}\right) - (1-a)\dfrac{\lambda}{n}$, $\forall a \in \left[\dfrac{1-\lambda}{2-\lambda}, 1\right)$.[5]
Then, choosing $a = \dfrac{1}{2} \in \left[\dfrac{1-\lambda}{2-\lambda}, 1\right)$, we get

$$Du \in L^{\frac{4n}{n-2\lambda}}(\Omega, \mathbb{R}^{nN})$$

and

$$|Du|_{0,\frac{4n}{n-2\lambda},\Omega} \le c_1 |D^2 u|^{\frac{1}{2}}_{0,2,\Omega} [u]^{\frac{1}{2}}_{\lambda,\Omega} + c_2 [u]_{\lambda,\Omega}$$

with $\frac{4n}{n-2\lambda} > 4$, therefore $Du \in L^4(\Omega, \mathbb{R}^{nN})$.

Now, if an inequality analogous to (9) is true for $u \in H^{1+\vartheta}(\Omega, \mathbb{R}^N) \cap C^{0,\lambda}(\Omega, \mathbb{R}^N)$, with $\vartheta, \lambda \in]0, 1[$, then, for ϑ and a suitably chosen, we could also deduce from this estimate that

$$Du \in L^4(\Omega, \mathbb{R}^{nN})$$

and hence to obtain, thanks also to the differentiability result (3), for the solutions $u \in L^2(-T, 0, H^1(\Omega, \mathbb{R}^N)) \bigcap C^{0,\lambda}(\overline{Q}, \mathbb{R}^N)$ of system (2), the regularity result (8), from which we can derive (7).

Then, the problem of seeing if the Gagliardo–Nirenberg inequalities (9) are true in the generalized Sobolev spaces arises. We will present a positive answer to this question (see Theorem 2).

[4] $D^2 u = \{D_{ij} u\} = \{D_i D_j u\}$, $i, j = 1, 2, \ldots, n$; $[u]_{\lambda,\Omega} = \sup\limits_{\substack{x,y \in \overline{\Omega} \\ x \ne y}} \dfrac{\|u(x) - u(y)\|}{\|x - y\|^\lambda}$.

[5] Making use of this theorem, we proved in [9] that the solutions $u \in L^2(-T, 0, H^2(\Omega, \mathbb{R}^N)) \cap C^{0,\lambda}(\overline{Q}, \mathbb{R}^N)$ of system (1) belong to $H^1(-T, 0, L^2(\Omega, \mathbb{R}^N))$ and, hence, the partial Hölder continuity of $D_i u$, $i = 1, 2, \ldots, n$, in Q.

Remark 1 Condition (8) is guaranteed by Naumann [11], for $u \in L^2(-T, 0, H^1(\Omega, \mathbb{R}^N)) \cap C^{0,\lambda}(\overline{Q}, \mathbb{R}^N)$ solution to system (2), provided that $\lambda \in]\frac{1}{2}, 1[$.

Remark 2 In the paper [12] the authors deal with the parabolic system

$$-\sum_{i=1}^{n} D_i a^i(Du) + \frac{\partial u}{\partial t} = B^0(Du)$$

and show that if $u \in L^2(-T, 0, H^1(\Omega, \mathbb{R}^N)) \cap C^{0,\lambda}(\overline{Q}, \mathbb{R}^N)$, $0 < \lambda < 1$, is a weak solution of this system, then

$$Du \in L^{\sigma}_{loc}(Q, \mathbb{R}^{nN})$$

with

$$4 \le \sigma < 4\left(1 + \frac{\lambda}{n}\right).$$

Remark 3 We recall that the partial-Hölder continuity of the solutions $u \in L^2(-T, 0, H^1(\Omega, \mathbb{R}^N)) \cap L^{\infty}(Q, \mathbb{R}^N)$ to system (1), under the so-called "smallness condition"

$$2M \sup_Q \|u\| < \nu$$

has been proved in the paper [4] (see also [5–7]).

Remark 4 In [8], the authors prove the interior differentiability result (7) assuming that $a^i(X, u, p) \in C^1(\overline{Q} \times \mathbb{R}^N \times \mathbb{R}^{nN})$ and that some growth conditions on a^i, $\frac{\partial a^i}{\partial x_r}$, $\frac{\partial a^i}{\partial u_k}$, $\frac{\partial a^i}{\partial p^j_k}$, $i, j, r = 1, \ldots, n$, $k = 1, \ldots, N$, are fulfilled.

2 Gagliardo–Nirenberg estimates in generalized Sobolev spaces

We recall the following Gagliardo–Nirenberg result (see [10,13]).

Theorem 1 *Let N be a positive integer and let Ω be an open bounded subset of $\mathbb{R}^n$, with the cone property. If*

$$u \in W^{m,r}(\Omega, \mathbb{R}^N) \cap C^{s,\lambda}(\Omega, \mathbb{R}^N)$$

with m integer ≥ 2, $1 < r < \infty$, s integer ≥ 0, $0 < \lambda < 1$, $s < m - 1$, then, for each integer j with $s + \lambda < j < m$, there exist two constants c_1 and c_2 (depending on Ω, m, r, s, λ, j) such that

$$\max_{|\alpha|=j} |D^{\alpha}u|_{0,p,\Omega} \le c_1 \left(\max_{|\alpha|=m} |D^{\alpha}u|_{0,r,\Omega}\right)^a \left(\max_{|\alpha|=s} [D^{\alpha}u]_{\lambda,\Omega}\right)^{1-a} + c_2 \max_{|\alpha|=s} [D^{\alpha}u]_{\lambda,\Omega},$$

where $\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) - (1-a)\frac{s+\lambda}{n}$, $\forall a \in \left[\frac{j-(s+\lambda)}{m-(s+\lambda)}, 1\right)$.

This interpolation result has been extended by Marino and Maugeri in the paper [8] to the case of generalized Sobolev spaces. More precisely, the authors proved the following result:

Theorem 2 *Let N be a positive integer and let* Ω *be an open bounded subset of* $\mathbb{R}^n$, *with the cone property. If*

$$u \in W^{m+\vartheta,r}(\Omega, \mathbb{R}^N) \cap C^{s,\lambda}(\Omega, \mathbb{R}^N)$$

with m *integer* ≥ 1, $0 < \vartheta < 1$, $1 < r < \infty$, s *integer* ≥ 0, $0 < \lambda < 1$, $s < m$, *then, for each* j *with* $\max\{s+\lambda, m+\vartheta-\frac{n}{r}\} < j < m+\vartheta$, *one has*

$$u \in W^{j,p}(\Omega, \mathbb{R}^N)$$

and there exists a constant c (*depending on* Ω, m, ϑ, r, s, λ, j, n, a) *such that*

$$\|u\|_{j,p,\Omega} \leq c\|u\|^{a}_{m+\vartheta,r,\Omega}\|u\|^{1-a}_{C^{s,\lambda}(\Omega,\mathbb{R}^N)},$$

where $\dfrac{1}{p} = \dfrac{j}{n} + a\left(\dfrac{1}{r} - \dfrac{m+\vartheta}{n}\right) - (1-a)\dfrac{s+\lambda}{n}$, $\forall a \in \left[\dfrac{j-(s+\lambda)}{m+\vartheta-(s+\lambda)}, 1\right)$ *with* $(1-a)(s+\lambda) + a(m+\vartheta)$ *noninteger.*

As a special case of Theorem 2 one gets:

Corollary 1 *Let N be a positive integer and let* Ω *be an open bounded subset of* $\mathbb{R}^n$, *with the cone property. If*

$$u \in H^{1+\vartheta}(\Omega, \mathbb{R}^N) \cap C^{0,\lambda}(\Omega, \mathbb{R}^N)$$

with $0 < \vartheta < 1$ *and* $0 < \lambda < 1$, *then*

$$u \in W^{1,p}(\Omega, \mathbb{R}^N)$$

and there exists a constant c (*depending on* Ω, ϑ, λ, n, a) *such that*

$$\|u\|_{1,p,\Omega} \leq c\|u\|^{a}_{1+\vartheta,\Omega}\|u\|^{1-a}_{C^{0,\lambda}(\Omega,\mathbb{R}^N)},$$

where $\dfrac{1}{p} = \dfrac{1}{n} + a\left(\dfrac{1}{2} - \dfrac{1+\vartheta}{n}\right) - (1-a)\dfrac{\lambda}{n}$, $\forall a \in \left(\dfrac{1-\lambda}{1+\vartheta-\lambda}, 1\right)$.

In particular, if $1-\lambda < \vartheta < 1$, *then setting* $a = \frac{1}{2}$ *we get:*

$$u \in W^{1,p}(\Omega, \mathbb{R}^N)$$

and there exists a constant c (*depending on* Ω, ϑ, λ, n) *such that*

$$\|u\|_{1,p,\Omega} \leq c\|u\|^{\frac{1}{2}}_{1+\vartheta,\Omega}\|u\|^{\frac{1}{2}}_{C^{0,\lambda}(\Omega,\mathbb{R}^N)},$$

where $p = 4 + \dfrac{8(\vartheta+\lambda-1)}{n-2(\vartheta+\lambda-1)} (> 4)$.

For the reader's convenience we briefly recall the proof of Theorem 2 (see [8] for details). It is well known that ([14,15]) the generalized Sobolev space $W^{m+\vartheta,r}(\Omega, \mathbb{R}^N)$ is equal to the Besov space $B^{m+\vartheta}_{r,r}(\Omega, \mathbb{R}^N)$ and that

$$C^{s,\lambda}(\Omega, \mathbb{R}^N) = B^{s+\lambda}_{\infty,\infty}(\Omega, \mathbb{R}^N).$$

Hence, the hypothesis $u \in W^{m+\vartheta,r}(\Omega, \mathbb{R}^N) \cap C^{s,\lambda}(\Omega, \mathbb{R}^N)$ is equivalent to

$$u \in B^{m+\vartheta}_{r,r}(\Omega, \mathbb{R}^N) \cap B^{s+\lambda}_{\infty,\infty}(\Omega, \mathbb{R}^N). \tag{10}$$

Now, for each $a \in (0, 1)$, we have

$$B^{m+\vartheta}_{r,r}(\Omega, \mathbb{R}^N) \cap B^{s+\lambda}_{\infty,\infty}(\Omega, \mathbb{R}^N) \subseteq \left(B^{s+\lambda}_{\infty,\infty}(\Omega, \mathbb{R}^N), B^{m+\vartheta}_{r,r}(\Omega, \mathbb{R}^N)\right)_a = B^h_{q,q}(\Omega, \mathbb{R}^N) \tag{11}$$

and

$$\|v\|_{B^h_{q,q}(\Omega,\mathbb{R}^N)} \le c\|v\|^a_{B^{m+\vartheta}_{r,r}(\Omega,\mathbb{R}^N)} \|v\|^{1-a}_{B^{s+\lambda}_{\infty,\infty}(\Omega,\mathbb{R}^N)}, \quad \forall v \in B^{m+\vartheta}_{r,r}(\Omega, \mathbb{R}^N) \cap B^{s+\lambda}_{\infty,\infty}(\Omega, \mathbb{R}^N), \tag{12}$$

where $h = (1 - a)(s + \lambda) + a(m + \vartheta)$, $\dfrac{1}{q} = \dfrac{a}{r}$.

In particular, from (10)–(12), it follows that

$$u \in B^h_{q,q}(\Omega, \mathbb{R}^N) \tag{13}$$

and that

$$\|u\|_{B^h_{q,q}(\Omega,\mathbb{R}^N)} \le c\|u\|^a_{B^{m+\vartheta}_{r,r}(\Omega,\mathbb{R}^N)} \|u\|^{1-a}_{B^{s+\lambda}_{\infty,\infty}(\Omega,\mathbb{R}^N)}. \tag{14}$$

Next, fix $j \in \left(\max\left\{s + \lambda, m + \vartheta - \dfrac{n}{r}\right\}, m + \vartheta\right)$, and consider (13) and (14) for $a \in \left[\dfrac{j - (s + \lambda)}{m + \vartheta - (s + \lambda)}, 1\right)$ with $(1 - a)(s + \lambda) + a(m + \vartheta)$ noninteger. For such values of a, setting $\dfrac{1}{p} = \dfrac{j}{n} + a\left(\dfrac{1}{r} - \dfrac{m + \vartheta}{n}\right) - (1 - a)\dfrac{s + \lambda}{n}$, it results

$$1 < q \le p < \infty, \quad 0 < j \le h < \infty, \quad h - \frac{n}{q} = j - \frac{n}{p}$$

and

$$B^h_{q,q}(\Omega, \mathbb{R}^N) = W^{h,q}(\Omega, \mathbb{R}^N) \subset W^{j,p}(\Omega, \mathbb{R}^N) \tag{15}$$

with algebraic and topological inclusion. From (13)–(15) the assertion follows.

3 Differentiability of the solutions to system (1)

Let $u \in L^2(-T, 0, H^1(\Omega, \mathbb{R}^N)) \bigcap C^{0,\lambda}(\overline{Q}, \mathbb{R}^N)$, $0 < \lambda < 1$, be a solution to system (1). Suppose that assumptions *(i)*, *(ii)*, and *(iii)* are fulfilled. From Theorem 2.III of [3], $\forall B(\rho) \subset\subset B(\sigma_0) \subset\subset \Omega$, $\forall b, b^* \in (0, T)$, with $b^* < b$, we obtain

$$u \in L^2(-b, 0, H^{1+\vartheta}(B(\sigma_0), \mathbb{R}^N)) \cap C^{0,\lambda}(\overline{Q}, \mathbb{R}^N), \quad \forall \vartheta \in (0, 1)$$

and the following estimate holds

$$\int_{-b^*}^0 |Du|^2_{\vartheta, B(\rho)} \mathrm{d}t \le c(\nu, K, U, \vartheta, \lambda, \rho, \sigma_0, b, b^*, n) \left\{1 + \int_{-b}^0 |u|^2_{1, B(\sigma_0)} \mathrm{d}t\right\}.^{6} \tag{16}$$

[6] $K = \sup_Q \|u\|$, $U = [u]_{\lambda,\overline{Q}} = \sup_{\substack{X,Y\in\overline{Q}\\ X\neq Y}} \dfrac{\|u(X) - u(Y)\|}{d^\lambda(X, Y)}$, where $d(X, Y)$ is the parabolic metric:

$$d(X, Y) = \max\left\{\|x - y\|, |t - \tau|^{1/2}\right\}, \quad X = (x, t), \quad Y = (y, \tau).$$

Hence, for a.e. $t \in (-b, 0)$, $u(x, t)$ belongs to $H^{1+\vartheta}(B(\sigma_0), \mathbb{R}^N) \cap C^{0,\lambda}(B(\sigma_0), \mathbb{R}^N)$ with $\vartheta = 1 - \frac{\lambda}{2} \in (0, 1)$.

From Corollary 1 we get, for a.e. $t \in (-b, 0)$, $u(x, t) \in W^{1,4}(B(\sigma_0), \mathbb{R}^N)$ and, for each $\rho \in (0, \sigma_0]$, it results

$$\|u\|^4_{1,4,B(\rho)} \le c(\lambda, \rho, n)\|u\|^2_{1+\vartheta,B(\rho)}\|u\|^2_{C^{0,\lambda}(B(\rho),\mathbb{R}^N)}, \quad \vartheta = 1 - \frac{\lambda}{2}$$

from which it follows

$$\|u\|^4_{1,4,B(\rho)} \le c(K, U, \lambda, \rho, n)\left\{1 + |u|^2_{1,B(\rho)} + |Du|^2_{1-\frac{\lambda}{2},B(\rho)}\right\}. \tag{17}$$

Then we deduce

$$\int_{-b}^{0} \|u\|^4_{1,4,B(\sigma_0)}\mathrm{d}t \le c\|u\|^2_{C^{0,\lambda}(\overline{Q},\mathbb{R}^N)} \int_{-b}^{0} \|u\|^2_{1+\vartheta,B(\sigma_0)}\mathrm{d}t$$

namely $Du \in L^4(B(\sigma_0) \times (-b, 0), \mathbb{R}^{nN})$ (see (8)).

Now we are able to prove the following

Theorem 3 *If $u \in L^2(-T, 0, H^1(\Omega, \mathbb{R}^N)) \bigcap C^{0,\lambda}(\overline{Q}, \mathbb{R}^N)$, $0 < \lambda < 1$, is a solution to system (1), if assumptions (i), (ii), and (iii) hold, then, $\forall B(\sigma) \subset\subset B(\sigma_0) \subset\subset \Omega$, $\forall a, b \in (0, T)$, $a < b$, one has*

$$u \in L^2(-a, 0, H^2(B(\sigma), \mathbb{R}^N)) \cap H^1(-a, 0, L^2(B(\sigma), \mathbb{R}^N))$$

and the following estimate holds

$$\int_{-a}^{0}\left\{|u|^2_{2,B(\sigma)} + \left|\frac{\partial u}{\partial t}\right|^2_{0,B(\sigma)}\right\}\mathrm{d}t \le c(\nu, K, U, \lambda, \sigma, \sigma_0, a, b, n)\left\{1 + \int_{-b}^{0} |u|^2_{1,B(\sigma_0)}\mathrm{d}t\right\}.$$

We give an outline of the proof.[7] First, we recall the following two lemmas.

Lemma 1 *If $v \in L^p(-b^*, -c, H^{1,p}(B(\rho), \mathbb{R}^N))$, $0 \le c < b^*$, $1 \le p < +\infty$, then, for each $\tau \in (0, 1)$ and $|h| < (1 - \tau)\rho$, we have*

$$\int_{-b^*}^{-c}\mathrm{d}t \int_{B(\tau\rho)} \|\tau_{i,h}v\|^p\mathrm{d}x \le |h|^p \int_{-b^*}^{-c}\mathrm{d}t \int_{B(\rho)} \|D_i v\|^p\mathrm{d}x, \quad i = 1, 2, \ldots, n.$$

Lemma 2 *If $v \in L^2(-a, 0, L^2(B(\sigma_0), \mathbb{R}^{nN}))$, $a, \sigma, \sigma_0 > 0$, $\sigma < \sigma_0$, and if there exists a real number $M > 0$ such that*

$$\int_{-a}^{0}\mathrm{d}t \int_{B(\sigma)} \|\tau_{i,h}v\|^2\mathrm{d}x \le |h|^2 M, \quad \forall |h| < \sigma_0 - \sigma, \quad i = 1, 2, \ldots, n$$

then $v \in L^2(-a, 0, H^1(B(\sigma), \mathbb{R}^{nN}))$ and the following estimate holds

$$\int_{-a}^{0}\mathrm{d}t \int_{B(\sigma)} \|D_i v\|^2\mathrm{d}x \le M, \quad i = 1, 2, \ldots, n, \tag{18}$$

where $\tau_{i,h}v(x, t) = v(x + he^i, t) - v(x, t)$, $i = 1, 2, \ldots, n$, and $\{e^i\}_{i=1,2,\ldots,n}$ is the canonic base of $\mathbb{R}^n$.

[7] The elliptic version of this result is stated in [1].

Proof of Theorem 3 Fix $B(\sigma) \subset\subset B(\sigma_0) \subset\subset \Omega$, $a, b \in (0, T)$ with $a < b$, and $i \in \mathbb{N}$ with $1 \le i \le n$, set $b^* = \dfrac{a+b}{2}$ and denote by $\psi(x)$ and $\rho_m(t)$ the functions defined in (5) and (6). Then there exists a real number $h_0(\nu, K, U, \lambda, n) > 0$ such that, for each $h \in \mathbb{R}$, with $|h| < \min\left(h_0, \dfrac{\sigma_0 - \sigma}{8}\right)$, it results

$$\begin{aligned}
&\frac{\nu}{2}\int_{-b^*}^{-\frac{1}{m}} \mathrm{d}t \int_{B\left(\frac{\sigma+3\sigma_0}{4}\right)} \psi^2 \rho_m^2 \|\tau_{i,h} Du\|^2 \mathrm{d}x \\
&\le c(\nu, K, \sigma, \sigma_0, a, b, n)|h|^2 \int_{-b^*}^{-\frac{1}{m}} \mathrm{d}t \int_{B(\sigma_0)} \left(1 + \|Du\|^2\right) \mathrm{d}x \\
&\quad + c(\nu, K) \int_{-b^*}^{-\frac{1}{m}} \mathrm{d}t \int_{B\left(\frac{\sigma+3\sigma_0}{4}\right)} \psi^2 \rho_m^2 \|\tau_{i,h} u\|^2 \|Du\|^2 \mathrm{d}x \\
&\quad + c(K) \int_{-b^*}^{-\frac{1}{m}} \rho_m^2 \mathrm{d}t \int_{B\left(\frac{\sigma+7\sigma_0}{8}\right)} \left(1 + \|Du\|^2\right) \|\tau_{i,-h}(\psi^2 \tau_{i,h} u)\| \mathrm{d}x. \qquad (19)
\end{aligned}$$

In this estimate the crucial terms are the last two integrals of the right-hand side. The term

$$c(K) \int_{-b^*}^{-\frac{1}{m}} \rho_m^2 \mathrm{d}t \int_{B\left(\frac{\sigma+7\sigma_0}{8}\right)} \left(1 + \|Du\|^2\right) \|\tau_{i,-h}(\psi^2 \tau_{i,h} u)\| \mathrm{d}x$$

can be estimated in the following way:

$$\begin{aligned}
&c(K) \int_{-b^*}^{-\frac{1}{m}} \rho_m^2 \mathrm{d}t \int_{B\left(\frac{\sigma+7\sigma_0}{8}\right)} \left(1 + \|Du\|^2\right) \|\tau_{i,-h}(\psi^2 \tau_{i,h} u)\| \mathrm{d}x \\
&\le c(K) \left(\int_{-b^*}^{-\frac{1}{m}} \rho_m^2 \mathrm{d}t \int_{B\left(\frac{\sigma+7\sigma_0}{8}\right)} |h|^2 \left(1 + \|Du\|^2\right)^2 \mathrm{d}x \right)^{1/2} \\
&\quad \cdot \left(\int_{-b^*}^{-\frac{1}{m}} \rho_m^2 \mathrm{d}t \int_{B\left(\frac{\sigma+7\sigma_0}{8}\right)} |h|^{-2} \|\tau_{i,-h}(\psi^2 \tau_{i,h} u)\|^2 \mathrm{d}x \right)^{1/2} \qquad (20)
\end{aligned}$$

once one knows that $Du \in L^4\left(B\left(\dfrac{\sigma+7\sigma_0}{8}\right) \times \left(-b^*, -\dfrac{1}{m}\right), \mathbb{R}^{nN}\right)$.

We just proved this fact and from (20), making use of Lemma 1, we get, $\forall \varepsilon > 0$

$$\begin{aligned}
&c(K) \int_{-b^*}^{-\frac{1}{m}} \rho_m^2 \mathrm{d}t \int_{B\left(\frac{\sigma+7\sigma_0}{8}\right)} \left(1 + \|Du\|^2\right) \|\tau_{i,-h}(\psi^2 \tau_{i,h} u)\| \mathrm{d}x \\
&\le \frac{\varepsilon}{2} |h|^{-2} \int_{-b^*}^{-\frac{1}{m}} \rho_m^2 \mathrm{d}t \int_{B\left(\frac{\sigma+7\sigma_0}{8}\right)} \|\tau_{i,-h}(\psi^2 \tau_{i,h} u)\|^2 \mathrm{d}x \\
&\quad + c(K, \varepsilon)|h|^2 \int_{-b^*}^{-\frac{1}{m}} \rho_m^2 \mathrm{d}t \int_{B\left(\frac{\sigma+7\sigma_0}{8}\right)} \left(1 + \|Du\|^2\right)^2 \mathrm{d}x \\
&\le \varepsilon \int_{-b^*}^{-\frac{1}{m}} \rho_m^2 \mathrm{d}t \int_{B\left(\frac{\sigma+3\sigma_0}{4}\right)} \psi^2 \|\tau_{i,h} Du\|^2 \mathrm{d}x
\end{aligned}$$

$$+c(\sigma,\sigma_0,\varepsilon)|h|^2\int_{-b^*}^{-\frac{1}{m}}\rho_m^2\mathrm{d}t\int_{B\left(\frac{\sigma+7\sigma_0}{8}\right)}\|Du\|^2\mathrm{d}x$$

$$+c(K,\varepsilon)|h|^2\int_{-b^*}^{-\frac{1}{m}}\rho_m^2\mathrm{d}t\int_{B\left(\frac{\sigma+7\sigma_0}{8}\right)}\left(1+\|Du\|^4\right)\mathrm{d}x. \tag{21}$$

Setting $\varepsilon=\dfrac{\nu}{4}$, from (21) we deduce

$$c(K)\int_{-b^*}^{-\frac{1}{m}}\rho_m^2\mathrm{d}t\int_{B\left(\frac{\sigma+7\sigma_0}{8}\right)}\left(1+\|Du\|^2\right)\|\tau_{i,-h}(\psi^2\tau_{i,h}u)\|\mathrm{d}x$$
$$\le\frac{\nu}{4}\int_{-b^*}^{-\frac{1}{m}}\rho_m^2\mathrm{d}t\int_{B\left(\frac{\sigma+3\sigma_0}{4}\right)}\psi^2\|\tau_{i,h}Du\|^2\mathrm{d}x$$
$$+c(\nu,K,\sigma,\sigma_0,n)|h|^2\int_{-b^*}^{-\frac{1}{m}}\left(1+|u|^2_{1,B(\sigma_0)}+|u|^4_{1,4,B\left(\frac{\sigma+7\sigma_0}{8}\right)}\right)\mathrm{d}t$$

and, by virtue of (17)

$$c(K)\int_{-b^*}^{-\frac{1}{m}}\rho_m^2\mathrm{d}t\int_{B\left(\frac{\sigma+7\sigma_0}{8}\right)}\left(1+\|Du\|^2\right)\|\tau_{i,-h}(\psi^2\tau_{i,h}u)\|\mathrm{d}x$$
$$\le\frac{\nu}{4}\int_{-b^*}^{-\frac{1}{m}}\mathrm{d}t\int_{B\left(\frac{\sigma+3\sigma_0}{4}\right)}\psi^2\rho_m^2\|\tau_{i,h}Du\|^2\mathrm{d}x$$
$$+c(\nu,K,U,\lambda,\sigma,\sigma_0,n)|h|^2\int_{-b^*}^{-\frac{1}{m}}\left(1+|u|^2_{1,B(\sigma_0)}+|Du|^2_{1-\frac{\lambda}{2},B\left(\frac{\sigma+7\sigma_0}{8}\right)}\right)\mathrm{d}t, \tag{22}$$

which is the desidered estimate of the last term of the right-hand side of (19).

Using (22) and taking into account the meaning of ψ and ρ_m, (19) becomes

$$\int_{-b^*}^{-\frac{2}{m}}\mathrm{d}t\int_{B(\sigma)}\|\tau_{i,h}Du\|^2\mathrm{d}x\le c(\nu,K,U,\lambda,\sigma,\sigma_0,a,b,n)|h|^2$$
$$\cdot\int_{-b^*}^{0}\left(1+|u|^2_{1,B(\sigma_0)}+|Du|^2_{1-\frac{\lambda}{2},B\left(\frac{\sigma+7\sigma_0}{8}\right)}\right)\mathrm{d}t$$
$$+c(\nu,K)\int_{-b^*}^{0}\mathrm{d}t\int_{B\left(\frac{\sigma+3\sigma_0}{4}\right)}\|\tau_{i,h}u\|^2\|Du\|^2\mathrm{d}x,$$

from which, letting $m\to\infty$, yields

$$\int_{-a}^{0}\mathrm{d}t\int_{B(\sigma)}\|\tau_{i,h}Du\|^2\mathrm{d}x\le c(\nu,K,U,\lambda,\sigma,\sigma_0,a,b,n)|h|^2$$
$$\cdot\left\{1+\int_{-b^*}^{0}\left(|u|^2_{1,B(\sigma_0)}+|Du|^2_{1-\frac{\lambda}{2},B\left(\frac{\sigma+7\sigma_0}{8}\right)}\right)\mathrm{d}t\right\}$$
$$+c(\nu,K)\int_{-b^*}^{0}\mathrm{d}t\int_{B\left(\frac{\sigma+3\sigma_0}{4}\right)}\|\tau_{i,h}u\|^2\|Du\|^2\mathrm{d}x. \tag{23}$$

It remains to consider the integral

$$\int_{-b^*}^{0} \mathrm{d}t \int_{B\left(\frac{\sigma+3\sigma_0}{4}\right)} \|\tau_{i,h}u\|^2 \|Du\|^2 \mathrm{d}x.$$

Using the Hölder inequality and thanks to Lemma 1 (with $p = 4$) and (17), we obtain

$$\begin{aligned}
&\int_{-b^*}^{0} \mathrm{d}t \int_{B\left(\frac{\sigma+3\sigma_0}{4}\right)} \|\tau_{i,h}u\|^2 \|Du\|^2 \mathrm{d}x \\
&\le \left(\int_{-b^*}^{0} \mathrm{d}t \int_{B\left(\frac{\sigma+3\sigma_0}{4}\right)} \|\tau_{i,h}u\|^4 \mathrm{d}x\right)^{1/2} \left(\int_{-b^*}^{0} \mathrm{d}t \int_{B\left(\frac{\sigma+3\sigma_0}{4}\right)} \|Du\|^4 \mathrm{d}x\right)^{1/2} \\
&\le |h|^2 \left(\int_{-b^*}^{0} \mathrm{d}t \int_{B\left(\frac{\sigma+7\sigma_0}{8}\right)} \|Du\|^4 \mathrm{d}x\right)^{1/2} \left(\int_{-b^*}^{0} \mathrm{d}t \int_{B\left(\frac{\sigma+3\sigma_0}{4}\right)} \|Du\|^4 \mathrm{d}x\right)^{1/2} \\
&\le |h|^2 \int_{-b^*}^{0} |u|^4_{1,4,B\left(\frac{\sigma+7\sigma_0}{8}\right)} \mathrm{d}t \le c(K, U, \lambda, \sigma, \sigma_0, n)|h|^2 \\
&\quad \cdot \int_{-b^*}^{0} \left(1 + |u|^2_{1,B\left(\frac{\sigma+7\sigma_0}{8}\right)} + |Du|^2_{1-\frac{\lambda}{2},B\left(\frac{\sigma+7\sigma_0}{8}\right)}\right) \mathrm{d}t. \qquad (24)
\end{aligned}$$

Now, from (23) and (24), we get

$$\int_{-a}^{0} \mathrm{d}t \int_{B(\sigma)} \|\tau_{i,h}Du\|^2 \mathrm{d}x \le c(\nu, K, U, \lambda, \sigma, \sigma_0, a, b, n)|h|^2$$

$$\cdot \left\{1 + \int_{-b^*}^{0} \left(|u|^2_{1,B(\sigma_0)} + |Du|^2_{1-\frac{\lambda}{2},B\left(\frac{\sigma+7\sigma_0}{8}\right)}\right) \mathrm{d}t\right\}$$

from which, by (16) (with $\vartheta = 1 - \frac{\lambda}{2}$ and $\rho = \frac{\sigma+7\sigma_0}{8}$), we deduce, for each integer i, $1 \le i \le n$, and each $|h| < \min\left(h_0, \frac{\sigma_0-\sigma}{8}\right)$,

$$\int_{-a}^{0} \mathrm{d}t \int_{B(\sigma)} \|\tau_{i,h}Du\|^2 \mathrm{d}x \le c(\nu, K, U, \lambda, \sigma, \sigma_0, a, b, n)|h|^2 \left\{1 + \int_{-b}^{0} |u|^2_{1,B(\sigma_0)} \mathrm{d}t\right\}.$$

The last estimate (obviously true also for $\min\left(h_0, \frac{\sigma_0-\sigma}{8}\right) \le |h| < \sigma_0 - \sigma$), combined with Nirenberg's Lemma 2, provide

$$Du \in L^2(-a, 0, H^1(B(\sigma), \mathbb{R}^{nN}))$$

and

$$\int_{-a}^{0} |u|^2_{2,B(\sigma)} \mathrm{d}t \le c(\nu, K, U, \lambda, \sigma, \sigma_0, a, b, n) \left\{1 + \int_{-b}^{0} |u|^2_{1,B(\sigma_0)} \mathrm{d}t\right\}.$$

It remains to show that $u \in H^1(-a, 0, L^2(B(\sigma), \mathbb{R}^N))$ and the following estimate holds:

$$\int_{-a}^{0} \mathrm{d}t \int_{B(\sigma)} \left\|\frac{\partial u}{\partial t}\right\|^2 \mathrm{d}x \le c \left\{1 + \int_{-b}^{0} |u|^2_{1,B(\sigma_0)} \mathrm{d}t\right\}. \qquad (25)$$

Taking into account that (see [3], n. 3) $B^0(X,u,Du), D_i a^i(X,u,Du) \in L^2(B(\sigma) \times (-a,0), \mathbb{R}^N)$, $i = 1,2,\ldots,n$, and that, for each $\varphi \in C_0^\infty(B(\sigma) \times (-a,0), \mathbb{R}^N)$, one has

$$\int_{-a}^{0} \mathrm{d}t \int_{B(\sigma)} \left(u \middle| \frac{\partial \varphi}{\partial t}\right) \mathrm{d}x = -\int_{-a}^{0} \mathrm{d}t \int_{B(\sigma)} \left(B^0(X,u,Du) + \sum_{i=1}^{n} D_i a^i(X,u,Du) \middle| \varphi\right) \mathrm{d}x$$

we get

$$\frac{\partial u}{\partial t} \in L^2(B(\sigma) \times (-a,0), \mathbb{R}^N).$$

Finally, estimate (25) is true by virtue of the inequalities

$$\int_{-a}^{0} \mathrm{d}t \int_{B(\sigma)} \|B^0(X,u,Du)\|^2 \mathrm{d}x \le c(K) \int_{-a}^{0} \mathrm{d}t \int_{B(\sigma)} \left(1 + \|Du\|^4\right) \mathrm{d}x,$$

$$\int_{-a}^{0} \mathrm{d}t \int_{B(\sigma)} \sum_{i=1}^{n} \|D_i a^i(X,u,Du)\|^2 \mathrm{d}x \le c(\nu, K, \sigma, \sigma_0, a, b, n)$$

$$\cdot \left\{1 + \int_{-b^*}^{0} \mathrm{d}t \int_{B\left(\frac{\sigma+\sigma_0}{2}\right)} \|Du\|^4 \mathrm{d}x\right\}$$

and thanks to (17) and (16). □

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